Universality and Solovay-Kitaev Theorem

Seminar Quantum Algorithms

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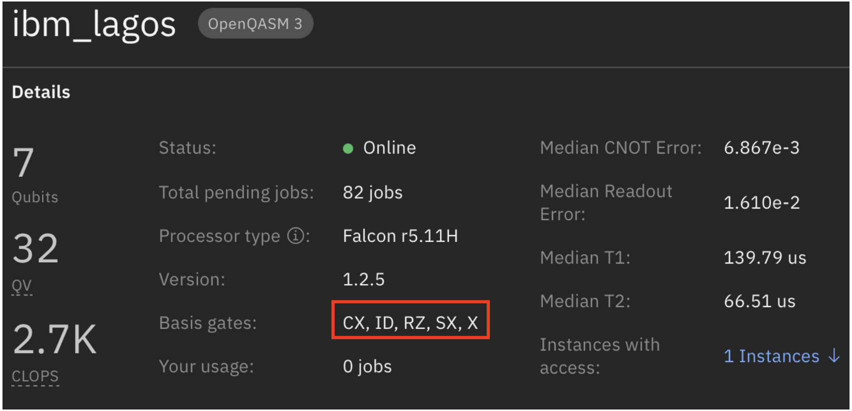
Motivation



Can we compute Quantum Circuits with small set of Basis gates?

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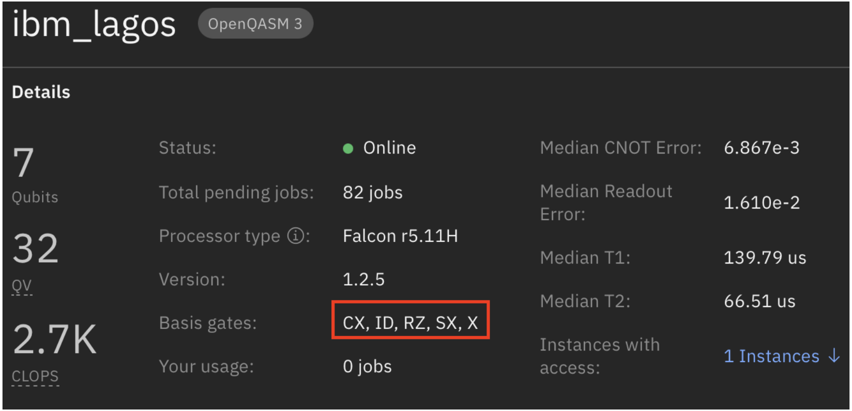

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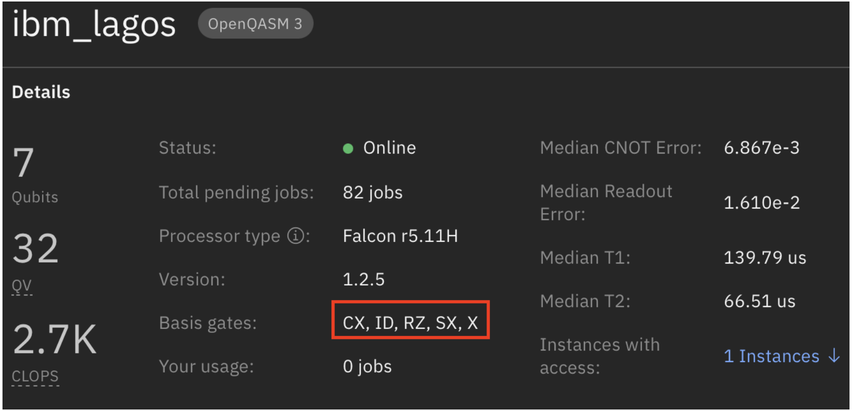

Can we compute Quantum Circuits with small set of Basis gates?
Can we compute efficiently with this set of Basis gates?
Is the complexity of quantum algorithms dependent on supported Basis gates?

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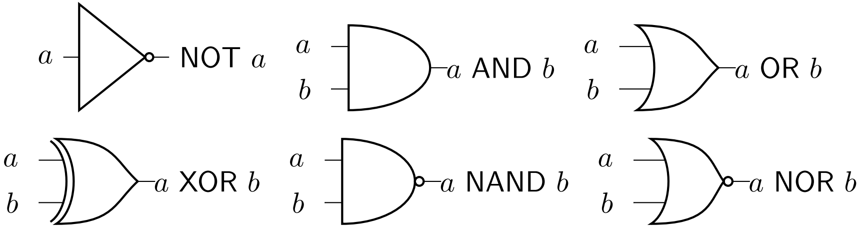
Elementary gates



Every gate that we can think of can be described by a truth table

Universality

- Claim: There exists a universal gate set s.t. we can compute every function $f : \{0,1\}^n \rightarrow \{0,1\}^m \Rightarrow$ Proof by induction

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- Consider $f : \{0,1\}^n \to \{0,1\}$
- $n = 1$: Four possible functions (truth table)
- $f_0(x_1, \ldots, x_n) \equiv f(0, x_1, \ldots, x_n), f_1(x_1, \ldots, x_n) \equiv f(1, x_1, \ldots, x_n)$
- $f(x) = (\overline{x_0} \cdot f_0(x_1, \ldots, x_n)) \oplus (x_0 \cdot f_1(x_1, \ldots, x_n))$

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- NAND gate is universal (exercise)

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Goal

Express an arbitrary n-Qubit gate with a sequence of 1-Qubit gates and CNOT

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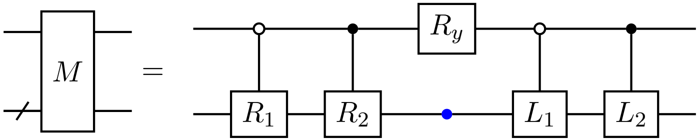
Arbitrary Unitary Gates

- Fact: M can be represented as $M = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} \cdot \begin{pmatrix} C & S \\ -S & C \end{pmatrix} \cdot \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$ where C, S are diagonal matrices with real entries and $C^2 + S^2 = I$



- Uniformly Controlled Rotation can be implemented with CNOT and rotation gates

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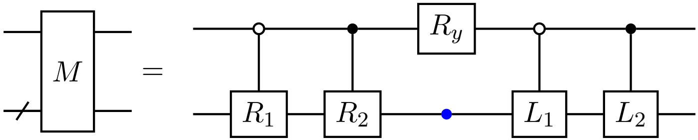


- Uniformly Controlled Rotation can be implemented with CNOT and rotation gates

$\Rightarrow$ n-Qubit gates can be expressed as a sequence of controlled gates, CNOT gates, and rotation gates

Goal

Express controlled gates as a sequence of 1-Qubit gates and CNOT

1-Qubit Gates

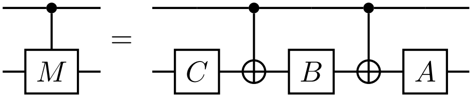
$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}$$

- arbitrary unitary 2×2 matrix only differs by global phase shift (exercise)
- every matrix $M \in \mathrm{SU}(2)$ can be represented as $M = R_z(\alpha) \cdot R_y(\theta) \cdot R_z(\beta)$

Controlled Gates

Lemma

For any $M \in \mathrm{SU}(2)$, *there exist matrices* A, B, C *s.t.* $A \cdot B \cdot C = I$ *and* $A \cdot X \cdot B \cdot X \cdot C = M$.



- for arbitrary unitary 2×2 matrix additional controlled phase gate (relative phase shift)

Controlled Gates - Proof

$$M = R_z(\alpha) \cdot R_y(\theta) \cdot R_z(\beta)$$

Set $A = R_z(\alpha) \cdot R_y\left(\frac{\theta}{2}\right)$, $B = R_y\left(-\frac{\theta}{2}\right) \cdot R_z\left(-\frac{\alpha+\beta}{2}\right)$, $C = R_z\left(\frac{\beta-\alpha}{2}\right)$.

$$A \cdot B \cdot C = R_z(\alpha) \cdot R_y\left(\frac{\theta}{2}\right) \cdot R_y\left(-\frac{\theta}{2}\right) \cdot R_z\left(-\frac{\alpha+\beta}{2}\right) \cdot R_z\left(\frac{\beta-\alpha}{2}\right)$$

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$$\begin{aligned} A \cdot B \cdot C &= R_z(\alpha) \cdot R_y\left(\frac{\theta}{2}\right) \cdot R_y\left(-\frac{\theta}{2}\right) \cdot R_z\left(-\frac{\alpha+\beta}{2}\right) \cdot R_z\left(\frac{\beta-\alpha}{2}\right) \\ &= R_z(\alpha) \cdot R_z(-\alpha) \end{aligned}$$

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$$AXBXC = R_z(\alpha) \cdot R_y\left(\frac{\theta}{2}\right) \cdot X \cdot R_y\left(-\frac{\theta}{2}\right) \cdot R_z\left(-\frac{\alpha+\beta}{2}\right) \cdot X \cdot R_z\left(\frac{\beta-\alpha}{2}\right)$$

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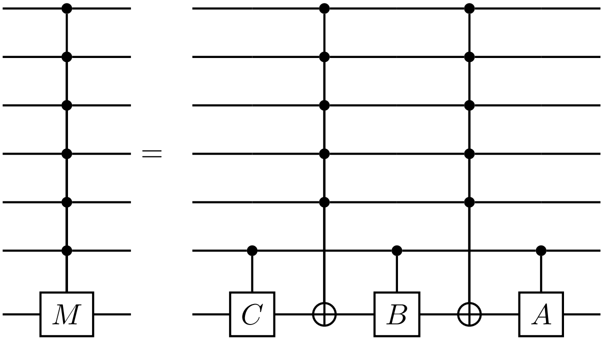
&= M
\end{aligned}$$

Controlled Gates II

$M \in \mathrm{SU}(2)$

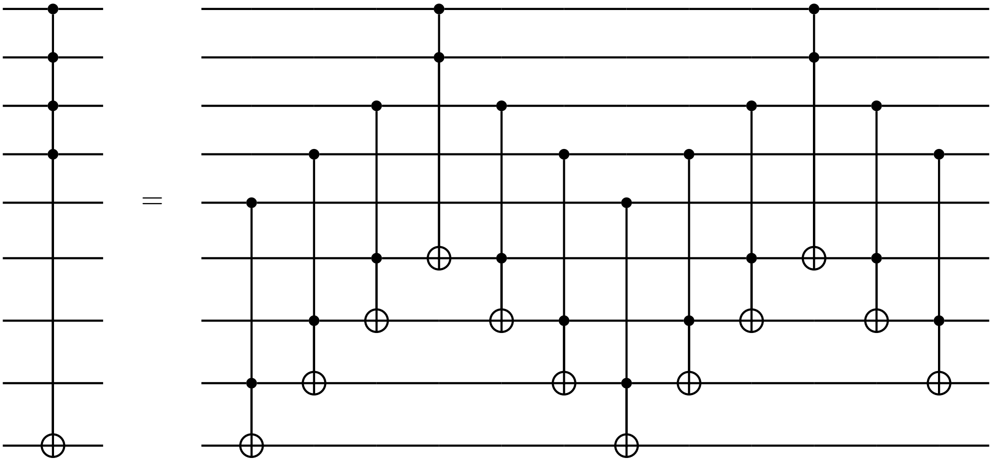


$\Rightarrow$ Multiple-Controlled gates can be realized with Multiple-Controlled Toffoli gates

Goal

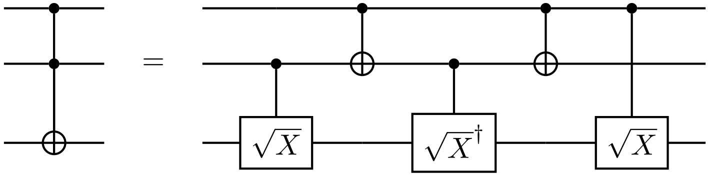
Express Multiple-Controlled Toffoli gates as a sequence of 1-Qubit gates and CNOT

Multiple-Controlled Toffoli gates



$\Rightarrow$ Multiple-Controlled Toffoli gates can be realized with Toffoli gates

Toffoli gates



$\Rightarrow$ Toffoli gate can be realized with controlled 1-Qubit gates and CNOT. Controlled 1-Qubit gates can be realized using previous lemma.

Quick Recap

- Arbitray Unitary $\Rightarrow$ Controlled gates, 1-Qubit gates, and CNOT via CSD

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$\Rightarrow$ Solovay-Kitaev

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Informal

Given an appropriate subset of $\mathrm{SU}(2)$, we can efficiently approximate every possible element in $\mathrm{SU}(2)$ arbitrarily well.

History Overview

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- **2010s** Results on most efficient compilation for specific sets
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- **2017** Bouland, Ozols: Property can be generalized to any gate set which contains an irreducible representation of a finite group.

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2017 Bouland, Ozols: Property can be generalized to any gate set which contains an irreducible representation of a finite group.

2020 Oszmaniec, Sawicki, Horodecki: Non-constructive inverse-free Solovay-Kitaev using results about spectral gaps of random walks on compact groups

2021 Bouland, Giurgica-Tiron: Constructive inverse-free Solovay-Kitaev

Informal

Given an appropriate subset of $\mathrm{SU}(2)$, we can efficiently approximate every possible element in $\mathrm{SU}(2)$ arbitrarily well.

Useful definitions - metric spaces

Let (X, d) be a metric space.

Definition

Let $A, N \subset X$ where N ist finite and $\varepsilon > 0$. N is called ε-net for A if

$$\forall a \in A \; \exists p \in N : d(a, p) < \varepsilon$$

Example

$\{0, 1\}$ is a $2/3$-net for the interval $[0, 1]$ but not for the interval $[0, 2]$.

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Useful definitions - trace norm

Definition

$$||A|| := \mathrm{tr}\,|A| = \mathrm{tr}\sqrt{A^\dagger A}$$

is called the trace norm.

The metric induced by the trace norm is given by $d(A,B) := ||A-B||$ and satisfies the following properties:

- unitary invariance: $||UAV|| = ||A||$ for any unitaries U and V,
- triangle inequality: $||A+B|| \leq ||A|| + ||B||$,
- submultiplicativity: $||AB|| \leq ||A|| \cdot ||B||$

Informal

Given an appropriate subset of $\mathrm{SU}(2)$, we can efficiently approximate every possible element in $\mathrm{SU}(2)$ arbitrarily well.

Gate set

- Let $\mathcal{G} \subset \mathrm{SU}(2)$ be a gate set.
- For the proof of Solovay-Kitaev we need $\mathcal{G}$ to be closed under inverses or do we?
- Notation: $\mathcal{G}^\ell = \left\{ g_1^{\alpha_1} g_2^{\alpha_2} \dots g_\ell^{\alpha_\ell} \mid g_i \in \mathcal{G}, \alpha_i = \pm 1 \right\}$, $\langle \mathcal{G} \rangle := \mathcal{G}^0 \cup \mathcal{G}^1 \cup \mathcal{G}^2 \cup \dots$

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- Solovay-Kitaev: We assume that $\mathcal{G}$ is finite subset of $\mathrm{SU}(2)$ that is closed under inverses and $\langle \mathcal{G} \rangle$ is dense in $\mathrm{SU}(2)$.

Solovay-Kitaev Theorem

Theorem

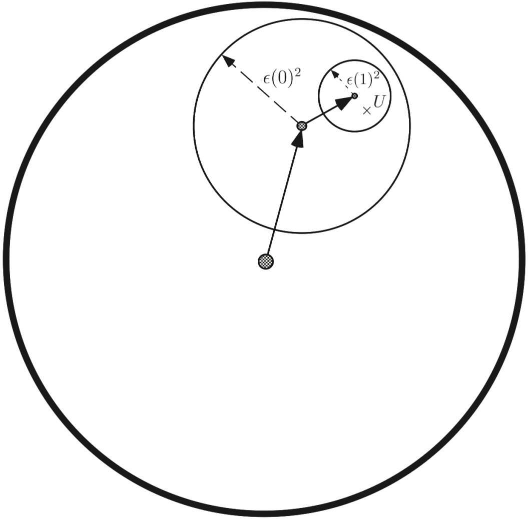
There is a constant c *s.t. for any* $\mathcal{G}$ *that is closed under inverses and* $\langle\mathcal{G}\rangle$ *is dense in* $\mathrm{SU}(2)$ *and* $\varepsilon > 0$ *one can choose* $\ell = \mathcal{O}(\log^c(1/\varepsilon))$ *so that* $\mathcal{G}^\ell$ *is an* ε*-net for* $\mathrm{SU}(2)$*. Furthermore, there exists an efficient algorithm that finds this approximation.*

In other words: The overhead of computing with a discrete universal gate set is poly-logarithmic.

Algorithm - Idea

Let $S_\varepsilon := \{U \in \mathrm{SU}(2) \mid ||U - I|| < \varepsilon\}$ be an open ε-ball in $\mathrm{SU}(2)$ around the identity

Construct series of ε-nets $\Gamma_0, \Gamma_1, \ldots$ s.t.



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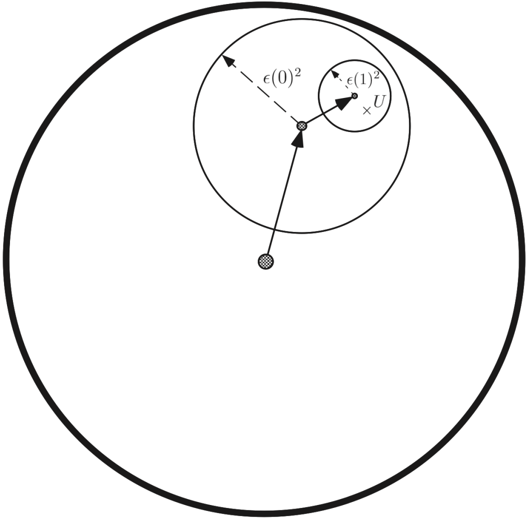

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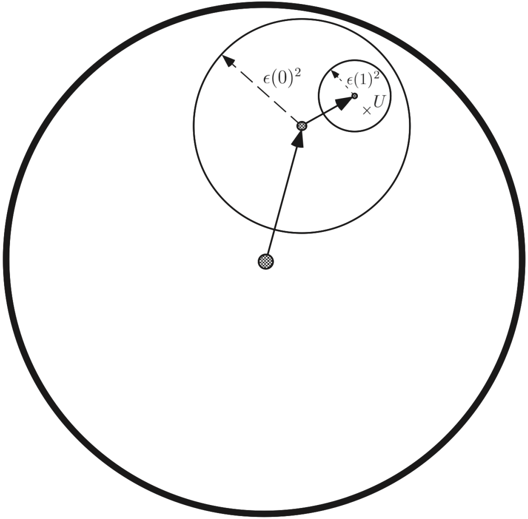
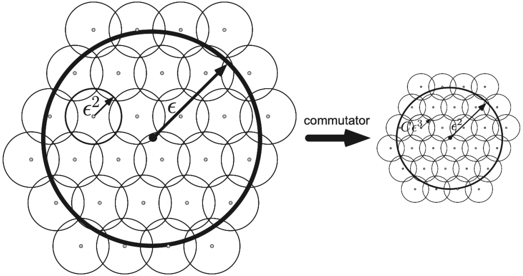
1. Start with initial approximation
2. Attack remaining distance with techniques that rely on being near the identity
3. Express precise matrices near the identity as strings of less precise matrices that are farther from the identity



Algorithm - Idea II

- Initial net Γ_0 can be created in constant time



Algorithm - Idea II

- Initial net Γ_0 can be created in constant time
- recursively: $\Gamma_k = [\![\Gamma_{k-1}, \Gamma_{k-1}]\!] := \{[\![A, B]\!] \mid A, B \in \Gamma_{k-1}\}$ where $[\![A, B]\!] = ABA^\dagger B^\dagger$ denotes the group commutator

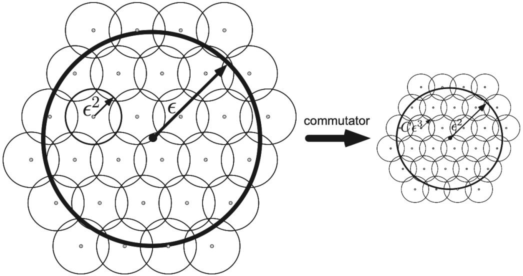


Figure: Taking group commutator of elements in S_ε fills in S_{ε^2} much more densely (Shrinking Lemma)

Shrinking Lemma

Lemma

There exist ε', s s.t. for any $\mathcal{G}$ and $\varepsilon \leq \varepsilon'$ we have: If $\mathcal{G}^{\ell}$ is an ε^2-net for S_ε then $\mathcal{G}^{5\ell}$ is an $s\varepsilon^3$-net for $S_{\sqrt{s\varepsilon^3}}$

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Corollary
There exist ε', s s.t. for any $\mathcal{G}, \varepsilon_0 \leq \varepsilon'$, and $k \in \mathbb{N}$ we have: If $\mathcal{G}^{\ell_0}$ is an ε^2-net for S_{ε_0} then $\mathcal{G}^{\ell_k}$ is an ε_k^2-net for S_{ε_k} where $\ell_k := 5^k \ell_0$ and $\varepsilon_k := (s\varepsilon_0)^{(3/2)^k}/s$.

Proof Solovay-Kitaev Idea

Theorem

There is a constant c s.t. for any $\mathcal{G}$ and $\varepsilon > 0$ one can choose $\ell = \mathcal{O}(\log^c(1/\varepsilon))$ so that $\mathcal{G}^\ell$ is an ε-net for $SU(2)$.

Corollary

There exist ε', s s.t. for any $\mathcal{G}, \varepsilon_0 \leq \varepsilon'$, and $k \in \mathbb{N}$ we have: If $\mathcal{G}^{\ell_0}$ is an ε^2-net for S_{ε_0} then $\mathcal{G}^{\ell_k}$ is an ε_k^2-net for S_{ε_k} where $\ell_k := 5^k \ell_0$ and $\varepsilon_k := (s\varepsilon_0)^{(3/2)^k}/s$.

The corollary allows to obtain good approximation for any element of $\mathrm{SU}(2)$ that is sufficiently close to identity. We now have to obtain a good approximation for any element of $\mathrm{SU}(2)$.
Start with rough approximation and use shrinking lemma.

Proof Solovay-Kitaev / Algorithm

1. Choose ε_0 wisely
2. $\langle \mathcal{G} \rangle$ dense in $\mathrm{SU}(2) \Rightarrow$ Choose ℓ_0 s.t. $\mathcal{G}^{\ell_0}$ is ε_0^2-net for $\mathrm{SU}(2)$.
3. Apply Shrinking Lemma repeatedly
4. Stop if given accuracy is reached

SK(U,n)

Input: $U \in \mathrm{SU}(2)$, depth n

Ouptut: $V \in \langle \mathcal{G} \rangle$ s.t. $||U - V|| < \varepsilon^2(n)$

if $n = 0$ **do**

$V = \varepsilon^2(0) - \text{APPROX}(U, G_I)$

else

$W = SK(U, n-1)$

$A, B = \text{FACTOR}(UW^\dagger)$

$V = [\![\text{SK}(A, n-1), \text{SK}(B, n-1)]\!] W$

Proof Solovay-Kitaev: Step 1

Choose ε_0 s.t.

- $\varepsilon_0 < \varepsilon'$ so that we can use Shrinking lemma
- $s\varepsilon_0 < 1$ so that (ε_k) decreases
- ε_0 small s.t. $\varepsilon_k^2 < \varepsilon_{k+1}$ so we can find closest current approximaton to our gate

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Proof Solovay-Kitaev: Step 2

$\langle \mathcal{G} \rangle$ dense in $\mathrm{SU}(2) \Rightarrow$ we can find ℓ_0 s.t. $\mathcal{G}^{\ell_0}$ is ε_0^2-net for $\mathrm{SU}(2)$

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Given $U \in \mathrm{SU}(2)$ we can choose $U_0 \in \mathcal{G}^{\ell_0}$ s.t. $||U - U_0|| < \varepsilon_0^2$.

Define $\Delta_1 := UU_0^\dagger$. Then:

$$||\Delta_1 - I|| = \left|\left|(U - U_0)U_0^\dagger\right|\right| = ||U - U_0|| < \varepsilon_0^2 < \varepsilon_1$$

$\Rightarrow \Delta_1 \in S_{\varepsilon_1}$

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Proof Solovay-Kitaev: Step 3

Shrinking Lemma $\Rightarrow \exists U_1 \in \mathcal{G}^{\ell_1}$ s.t.

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Define $\Delta_2 := \Delta_1 U_1^\dagger = UU_0^\dagger U_1^\dagger$. Then:

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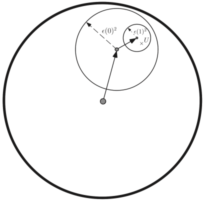

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$$\varepsilon_k^2 = \left((s\varepsilon_0)^{(3/2)^k} / s \right)^2 = \varepsilon$$

Solve for k:

$$\left(\frac{3}{2} \right)^k = \frac{\log(1/s^2 \varepsilon)}{2 \log(1/s\varepsilon_0)} = 5^{k/c}$$

for $c \approx 4$.

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To prove this lemma, we have to transform the parameters $(\ell, \varepsilon^2, \varepsilon) \mapsto (5\ell, s\varepsilon^3, \sqrt{s\varepsilon^3})$

Proof Shrinking Lemma

$(\ell, \varepsilon^2, \varepsilon) \mapsto (4\ell, s\varepsilon^3, \varepsilon^2) \mapsto (5\ell, s\varepsilon^3, \sqrt{s\varepsilon^3})$

Goal: Approximate U in S_{ε^2}

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$$\begin{tikzcd} V = e^{-iA}, W = e^{-iB} \arrow[r, "{[\![\cdot,\cdot]\!]}"] & {[\![V, W]\!]} \\ A, B \arrow[u] \arrow[r, "{[\cdot,\cdot]}"] & {[A, B]} \arrow[u, dotted, "?"] \end{tikzcd}$$

$$||A|| < \varepsilon, ||B|| < \varepsilon, \left|\left| [\![e^{-iA}, e^{-iB}]\!] - e^{-[A,B]} \right|\right| \leq \mathcal{O}(\varepsilon^3)$$

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Matrix commutator for $\mathrm{SU}(2)$: $V = u(\vec{a}) := e^{-\frac{i}{2}\vec{a}\cdot\vec{\sigma}}, W = u(\vec{b}) = e^{-\frac{i}{2}\vec{b}\cdot\vec{\sigma}}$ where
$\vec{r}\cdot\vec{\sigma} = r_x X + r_y Y + r_z Z$

$$[X, Y] = 2iZ, [Y, Z] = 2iX, [Z, X] = 2iY \Rightarrow [\vec{a}\cdot\vec{\sigma}, \vec{b}\cdot\vec{\sigma}] = 2i(\vec{a}\times\vec{b})\vec{\sigma}$$

$$u(\vec{a}\times\vec{b}) = e^{-\left[\frac{1}{2}\vec{a}\cdot\vec{\sigma}, \frac{1}{2}\vec{b}\cdot\vec{\sigma}\right]}$$

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$$u(\vec{a} \times \vec{b}) = e^{-\left[\frac{1}{2}\vec{a}\cdot\vec{\sigma}, \frac{1}{2}\vec{b}\cdot\vec{\sigma}\right]}$$

$$\Rightarrow \left|\left|[\![V, W]\!] - u(\vec{a} \times \vec{b})\right|\right| = \mathcal{O}(\varepsilon^3)$$

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Goal: Approximate $U = u(\vec{x})$ in $S_{\varepsilon^2}, |\vec{x}| < \varepsilon^2$
Main Idea:

- Write $\vec{x} = \vec{y} \times \vec{z}$ with $|\vec{y}|, |\vec{z}| \leq \varepsilon$
- Approximate $u(\vec{y}), u(\vec{z})$ with $\vec{y_0}, \vec{z_0}$ s.t. $u(\vec{y_0}), u(\vec{z_0}) \in \mathcal{G}^\ell$ is ε^2-approximation

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$$||u(\vec{x}) - [\![u(\vec{y_0}), u(\vec{z_0})]\!]|| \leq ||u(\vec{x}) - u(\vec{y_0} \times \vec{z_0})|| + ||u(\vec{y_0} \times \vec{z_0}) - [\![u(\vec{y_0}), u(\vec{z_0})]\!]|| \leq s\varepsilon^3$$

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$\Rightarrow [\![u(\vec{y_0}), u(\vec{z_0})]\!]$ $s\varepsilon^3$-approximates U in 4ℓ gates $\Rightarrow s\varepsilon^3$-net for S_{ε^2}
Now: Perform translation step: Given $U \in S_{\sqrt{s\varepsilon^3}}$ we can find $V \in \mathcal{G}^\ell$ s.t.
$||U - V|| \leq \varepsilon^2 \Rightarrow UV^\dagger \in S_{\varepsilon^2}$
Find $W_1, W_2 \in \mathcal{G}^\ell$ s.t. $||[\![W_1, W_2]\!] - UV^\dagger|| \leq s\varepsilon^3 \Rightarrow ||[\![W_1, W_2]\!]V - U|| \leq s\varepsilon^3$

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How to do in general?

Self-correcting sequences

Definition

Consider operators $\{g_1, \ldots, g_k\} \subset \mathrm{SU}(d)$ and set of corresponding ε-approximate operators $\{g'_1, \ldots, g'_k\} \subset \mathrm{SU}(d)$ s.t. $||g'_i - g_i|| \leq \varepsilon$. A self-correcting sequence is a word in the approximate operators which approximate the identity to a higher order in ε

$$g'_{i_1} \ldots g'_{i_N} = I + \mathcal{O}(\varepsilon^n) \quad n > 1$$

Bouland, Giurgica-Tiron (2021): There exists quadratically-precise sequence in $\mathrm{SU}(d)$

Bouland, Giurgica-Tiron

Use Pauli approximations

$$X' = X + \mathcal{O}(\varepsilon)$$
$$Z' = Z + \mathcal{O}(\varepsilon)$$

Dimension $d = 2$: $Z'X'X'Z'X'Z'Z'X' = I + \mathcal{O}(\varepsilon^2) \quad N = 8$

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Dimension $d \geq 2$: $\left(Z'X'^d\right)^{d-1} Z' \left(X'Z'^d\right)^{d-1} X' = I + \mathcal{O}(\varepsilon^2) \quad N = 2d^2$

How to invert U: We have $X' = X + \mathcal{O}(\varepsilon), Z' = Z + \mathcal{O}(\varepsilon), \hat{U}^\dagger = U^\dagger + \mathcal{O}(\varepsilon)$

Bouland, Giurgica-Tiron

Use Pauli approximations

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$X'\hat{U}^\dagger U = X + \mathcal{O}(\varepsilon)$

Let $J(X', Z') = I + \mathcal{O}(\varepsilon^2)$ be a self-correcting sequence in X', Z'.

$\Rightarrow J(X'\hat{U}^\dagger U, Z') = I + \mathcal{O}(\varepsilon^2)$

Bouland, Giurgica-Tiron

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How to invert U: We have $X' = X + \mathcal{O}(\varepsilon), Z' = Z + \mathcal{O}(\varepsilon), \hat{U}^\dagger = U^\dagger + \mathcal{O}(\varepsilon)$

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Let $J(X', Z') = I + \mathcal{O}(\varepsilon^2)$ be a self-correcting sequence in X', Z'.

$\Rightarrow J(X'\hat{U}^\dagger U, Z') = I + \mathcal{O}(\varepsilon^2)$

This sequence is close to identity and in an instance of U itself. Remove U and done ;)

Consequences & Open problems

Consequences:

- Sequence for inverses has length $\mathcal{O}(d^2)$
  $\Rightarrow \#(\text{gates}) = \mathcal{O}(\log^c(1/\varepsilon)), c = \mathcal{O}(\log d)$.
- simplifies proofs in various areas of quantum complexity theory
- Construction could be practically useful when errors are coherent e.g. in dynamic decoupling

Open problems:

- Reduce exponent from $\mathcal{O}(\log d)$ to the nonconstructive upper bound of 3.
- Understand mathematics of self-correcting sequences. Generalize to other groups and higher orders
- ...

Solovay-Kitaev Rap by ChatGPT

Yo, let me tell you about a theorem so neat
It's called the Solovay-Kitaev, let's take a seat
It's about quantum gates and approximation, you see
Making quantum computing even better, that's the key

Solovay-Kitaev, Solovay-Kitaev
Universal quantum gates, we can achieve
Polynomial complexity, that's the key
Approximation with precision, can't you see?

For any finite group G and positive ϵ
We can approximate any U-gate with precision
Using a finite set of quantum gates, we can't go wrong
Polylogarithmic complexity, won't take too long

Solovay-Kitaev, Solovay-Kitaev
Universal quantum gates, we can achieve
Polynomial complexity, that's the key
Approximation with precision, can't you see?

With Solovay-Kitaev, we can compute with ease
More complex operations, our limits will increase
Like a puzzle, we fit the gates to get the right solution
And quantum computing will become a revolution

Solovay-Kitaev, Solovay-Kitaev
Universal quantum gates, we can achieve
Polynomial complexity, that's the key
Approximation with precision, can't you see?

So let's celebrate Solovay-Kitaev, let's give it a cheer
For the future of quantum computing is looking so clear!

Thank You! Any Questions?